

$\mathcal{N} = 1$ extension of minimal model holography

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ABSTRACT: The CFT dual of the higher spin theory with minimal $\mathcal{N} = 1$ spectrum is determined. Unlike previous examples of minimal model holography, there is no free parameter beyond the central charge, and the CFT can be described in terms of a non-diagonal modular invariant of the bosonic theory at the special value of the 't Hooft parameter $\lambda = \frac{1}{2}$. As evidence in favour of the duality we show that the symmetry algebras as well as the partition functions agree between the two descriptions.

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1 Introduction

Recently, dualities relating higher spin theories on AdS spaces [1] to vector-like conformal field theories have attracted some attention. The original idea was already suggested some time ago [2–4] and first concrete proposals were made soon thereafter [5, 6], but it was only through the work of Giombi & Yin [7, 8] that compelling evidence was obtained. Dualities of this kind are very interesting because they hold the promise of offering insights into the conceptual underpinning of the AdS/CFT correspondence.

More recently, a lower dimensional version, relating higher spin theories on AdS_3 [9, 10] to the large N limit of some 2d minimal model CFTs was proposed [11]. These incarnations are interesting since higher spin theories in 3d can be much more easily described in terms of a Chern-Simons formulation. At the same time, 2d minimal model CFTs are under very good analytical control, and thus the correspondence can be analysed and tested in quite some detail, see e.g. [12–15] for the matching of the symmetries; [16–18] for the comparison of the spectrum; and [19–23] for the analysis of correlation functions. Finally, these models evade the Maldacena-Zhiboedov theorem [24, 25] that implies that higher dimensional theories with an unbroken higher spin symmetry and finitely many degrees of freedom are necessarily free. For a review of this circle of ideas see [26].

The original proposal of [11] has been generalised in a variety of directions: to the case with orthogonal gauge groups [27, 28], the situation with $\mathcal{N} = 2$ supersymmetry [29] (see also [30]), and more recently to the case with $\mathcal{N} = 1$ supersymmetry [31] (see also [32]). In this paper we give evidence for a different $\mathcal{N} = 1$ supersymmetric duality, for which the spin content of the higher spin gauge fields is minimal, i.e. it consists of a single higher spin field for each half-integer spin $s \geq \frac{3}{2}$. Unlike the previous examples, there is no additional free parameter (except for the central charge that is proportional to the radius of the AdS space) in this case: this can be seen from the AdS point of view where the corresponding higher spin algebra is the $\text{shs}(1|2)$ algebra of [33] that does not have a deformation parameter; it also follows from the analysis of the most general $s\mathcal{W}_\infty$ algebra with the above spin content for which we have found (see Section 4) that the Jacobi identities do not allow for a free coupling constant. In fact, as we shall explain in Section 3, the relevant CFT can be described as the $\mathcal{N} = 1$ supersymmetric extension of the bosonic theory of [11] that exists provided that the level of the numerator $\mathfrak{su}(N)_k$ algebra equals $k = N$. From this point of view, the $\mathcal{N} = 1$ supersymmetric theory then corresponds to a ‘non-diagonal’ modular invariant. Our duality is therefore the first interesting example where one of the non-diagonal coset modular invariants has been identified with a dual higher spin theory.

The paper is organised as follows. In Section 2 we review some of the salient features of the bosonic duality of [11] that will be relevant in the following. Section 3 explains how this theory can be extended for $k = N$ to an $\mathcal{N} = 1$ superconformal field theory. In particular, we calculate the extended characters and give an explicit formula for the full $\mathcal{N} = 1$ superconformal partition function, see eq. (3.21). In Section 4 we then analyse the most general $\mathcal{N} = 1$ superconformal $s\mathcal{W}$ -algebra with the given spectrum, and show that it does not possess any free parameter (except for the central charge). We also show that it contains the bosonic $\mathcal{W}_\infty[\frac{1}{2}]$ algebra as a subalgebra (see Section 4.5), as expected from the above construction. Section 5 describes the dual higher spin theory, and shows that the asymptotic symmetry algebra of the Chern-Simons theory based on $\text{shs}(1|2)$ agrees indeed with the ‘wedge’ algebra of our $s\mathcal{W}$ -algebra. Finally, we explain how the full spectrum of the coset CFT in the ’t Hooft limit can be accounted for by adding to the higher spin fields a complex $\mathcal{N} = 1$ matter multiplet, see Section 5.3. Section 6 contains a brief conclusion, and there are three appendices where some of the more technical material is explained.

2 Review of bosonic minimal model holography

Let us start by reviewing the bosonic duality of [11], see [26] for a recent review. The bosonic higher spin theory on AdS_3 based on the Lie algebra $\text{hs}[\lambda]$ [9, 10] is conjectured to be dual to the 't Hooft like large N limit of the cosets

$$\frac{\mathfrak{su}(N)_k \oplus \mathfrak{su}(N)_1}{\mathfrak{su}(N)_{k+1}} \quad (2.1)$$

with central charge

$$c = (N-1) \left(1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right), \quad (2.2)$$

where λ is identified with the 't Hooft coupling

$$\lambda = \frac{N}{N+k} \quad (2.3)$$

that is held fixed in the large N, k limit. In particular, the symmetries of the coset define a $\mathcal{W}_\infty[\mu]$ algebra that is generated by the stress energy tensor, together with one Virasoro primary field of each integer spin $s \geq 3$. For a given value of the central charge c , the $\mathcal{W}_\infty[\mu]$ algebras corresponding to three (generically different) values of μ describe isomorphic \mathcal{W} -algebras [15], and this ‘triality’ of relations explains, in particular, why the quantisation of the asymptotic symmetry algebra of the higher spin theory (that was first determined classically in [12–14]) agrees with that of the dual coset.

In addition to the vacuum representation (that just describes the $\mathcal{W}_\infty[\lambda]$ algebra), the coset theory contains a number of irreducible representations that are labelled by the pairs $(\Lambda_+; \Lambda_-)$, where Λ_+ and Λ_- are representation of $\mathfrak{su}(N)_k$ and $\mathfrak{su}(N)_{k+1}$, respectively. These degrees of freedom correspond, on the higher spin side, to those of a complex scalar field with mass

$$M^2 = -(1 - \lambda^2), \quad (2.4)$$

as well as a number of classical solutions [15, 17, 18], see also [34, 35] for a somewhat different interpretation. The spectrum of the coset theories is taken to be given by the ‘A-type modular invariant’ partition function, i.e. it is of the form

$$\mathcal{H} = \bigoplus_{(\Lambda_+, \Lambda_-)} (\Lambda_+; \Lambda_-) \otimes \overline{(\Lambda_+^*; \Lambda_-^*)}, \quad (2.5)$$

where the sum runs over all inequivalent coset representations, and Λ^* denotes the representation conjugate to Λ . The states in the representations $(\Lambda_+; 0)$ correspond to the excitations of the complex scalar field, while the remaining states account for the classical solutions (that are labelled by $(0; \Lambda_-)$), as well as its scalar excitations. In the strict 't Hooft limit, some of the latter states decouple, and the resulting partition function agrees precisely with the 1-loop thermal partition function of the higher spin theory with two complex scalar fields [16].

3 The minimal $\mathcal{N} = 1$ susy extension of the bosonic cosets

The above bosonic coset can be minimally extended to an $\mathcal{N} = 1$ superconformal algebra when $k = N$, i.e. when $\lambda = \frac{1}{2}$. Indeed, at $k = N$ the WZW model based on $\mathfrak{su}(N)_N$ has central charge $\frac{1}{2}(N^2 - 1)$, and can be realised in terms of $(N^2 - 1)$ free fermions. Thus there exists a conformal embedding

$$\mathfrak{su}(N)_N \hookrightarrow \mathfrak{so}(N^2 - 1)_1 , \quad (3.1)$$

and we can make the bosonic theory supersymmetric by considering the charge conjugation modular invariant based on $\mathfrak{so}(N^2 - 1)_1$ rather than $\mathfrak{su}(N)_N$. (From the point of view of the original coset, the resulting theory then corresponds to a ‘D-type modular invariant’.)

More specifically, the branching rule of the vacuum representation of $\mathfrak{so}(N^2 - 1)_1$ into $\mathfrak{su}(N)_N$ representations is of the form

$$\mathcal{H}_0^{(1)} = [0^{N-1}] \oplus [2, 0^{N-4}, 1, 0] \oplus [0, 1, 0^{N-4}, 2] \oplus \dots , \quad (3.2)$$

as was already shown in [36]. Similarly, for the vector representation of $\mathfrak{so}(N^2 - 1)_1$ the first few terms are

$$\mathcal{H}_v^{(1)} = [1, 0^{N-3}, 1] \oplus [1, 1, 0^{N-5}, 1, 1] \oplus \dots . \quad (3.3)$$

We shall momentarily explain how to describe the full decomposition series in both cases. Before we get to this we should also mention that $\mathfrak{so}(N^2 - 1)_1$ has two spinor representations, whose conformal weight however scales with N . They will therefore not play any role in the ’t Hooft limit, compare also the discussion in [28].

3.1 The branching functions

In order to identify the full branching rules describing (3.2) and (3.3) we recall that the conformal weight of the primary in the representation Π of $\mathfrak{su}(N)_N$ equals

$$h_{\text{WZW}}(\Pi) = \frac{C_2(\Pi)}{2N} , \quad (3.4)$$

where the quadratic Casimir takes the form

$$2 C_2(\Pi) = \langle \Pi, \Pi + 2\rho \rangle = |\Pi|N + \sum_{i=1}^{N-1} r_i^2 - \sum_j c_j^2 - \frac{|\Pi|^2}{N} . \quad (3.5)$$

Here $|\Pi|$ is the total number of boxes of the Young diagram corresponding to Π , r_j is the number of boxes in the j^{th} row, while c_j denotes the number of boxes in the j^{th} column. In the ’t Hooft limit, it is natural to think of Π as being given in terms of two finite subdiagrams Π_l and Π_r that denote the contributions of the boxes and anti-boxes, respectively. In terms of these, the quadratic Casimir has the form [37] (note that the different factor of 2 comes from a different normalisation of the quadratic Casimir relative to [37])

$$C_2(\Pi) = C_2(\Pi_l) + C_2(\Pi_r) + \frac{|\Pi_l||\Pi_r|}{N} . \quad (3.6)$$

This identity is actually true even at finite N , irrespective of how Π is split up into Π_l and Π_r , see the figure and explanation in [30]. The $\mathfrak{su}(N)_N$ representations that should be included in the vacuum or vector representations of $\mathfrak{so}(N^2 - 1)_1$ are those for which the conformal weight in (3.4) is an integer or a half-integer at finite N . (The vector representation of $\mathfrak{so}(N^2 - 1)_1$ has conformal dimension $\frac{1}{2}$.) Because of (3.4) and (3.5) it follows that this condition is satisfied if one can represent Π by a pair of subdiagrams Π_l and Π_r which are related to one another by transposition, i.e. if $\Pi_l = \Pi_r^t$. In the 't Hooft limit, these are the only such representations: indeed, representations that lead to integer or half-integer conformal weights have the property that $|\Pi_l| - |\Pi_r|$ is a multiple of N . The condition that both diagrams Π_l and Π_r remain finite in the limit then implies $|\Pi_l| = |\Pi_r|$. Thus it is enough to check that the order 1 term in (3.6) vanishes, and it is easy to see that this is only the case provided that the two representations are related to one another by transposition. In the following we shall denote this set of representations by

$$\Omega = \{ \Pi = (\Pi_l, \Pi_r) \mid \Pi_l = \Pi_r^t \} . \quad (3.7)$$

Note that the first few terms in (3.2) and (3.3) are indeed of this form.

Next we observe that all representations in Ω are in fact in the zeroth congruence class of $\mathfrak{su}(N)$, i.e. their Dynkin labels satisfy

$$l_1 + 2l_2 + \cdots + (N-1)l_{N-1} = |\Pi| \equiv |\Pi_l| - |\Pi_r| \equiv 0 \pmod{N} . \quad (3.8)$$

Thus the coset representations $(\Pi; 0)$ with $\Pi \in \Omega$ satisfy the coset selection rule with the representation of $\mathfrak{su}(N)_1$ being the trivial vacuum representation; hence it is consistent to add these representations to the vacuum representation (or to $([1, 0^{N-3}, 1]; 0)$). Depending on whether the number of boxes $|\Pi_l| = |\Pi_r|$ is even or odd, the conformal dimension is integer or half-integer, and hence Π contributes to the extended vacuum or vector representations of $\mathfrak{so}(N^2 - 1)_1$, respectively, see (3.2) and (3.3). In a fermionic theory it is natural to add both of them together; the sum of these two representations then defines the superconformal vacuum representation.

We can similarly define the extended representations

$$\mathcal{H}_\Lambda = \bigoplus_{\Pi \in \Omega} (\Pi; \Lambda) , \quad (3.9)$$

and by construction it is clear that the conformal weights of the various different representations differ again by integers or half-integers. On the other hand we cannot extend any of the representations of the form $(\Lambda_+; \Lambda_-)$ in any obvious manner, unless $\Lambda_+ = 0$ (or $\Lambda_+ \in \Omega$). Thus the full extended theory is of the form

$$\mathcal{H} = \bigoplus_{\Lambda} \mathcal{H}_\Lambda \otimes \overline{\mathcal{H}_{\Lambda^*}} = \bigoplus_{\Lambda} \bigoplus_{\Pi, \Pi' \in \Omega} (\Pi; \Lambda) \otimes \overline{(\Pi'; \Lambda^*)} , \quad (3.10)$$

where the first sum runs over all pairs $\Lambda = (\Lambda_l, \Lambda_r)$ of finite Young diagrams, and the conjugate representation equals $\Lambda^* = (\Lambda_r, \Lambda_l)$. This theory contains fermionic as well as bosonic fields (since the conformal weights associated to the representations in Ω can be

half-integer as well as integer), and the corresponding partition function is therefore not invariant under $T : \tau \mapsto \tau + 1$, but only under $T^2 : \tau \mapsto \tau + 2$. This can be cured in the usual manner by introducing a GSO-projection onto the bosonic states (as well as adding in the spinor representations). However, in order to relate the theory to the higher spin dual theory, it is more natural to consider directly this fermionic theory.

3.2 The extended characters and the spin spectrum

Next we want to study the partition function of (3.10). In particular, we want to show that the extended vacuum representation defines the minimal $\mathcal{N} = 1$ superconformal extension of $\mathcal{W}_\infty[\frac{1}{2}]$. We also need to evaluate the contributions of the other sectors in order to be able to identify them with suitable matter field contributions on the higher spin side.

Recall from [16] that in the 't Hooft limit the character of the bosonic coset representation $(\Pi; \Lambda)$ is of the form

$$b_{\Pi; \Lambda}^\lambda(q) = q^{-\frac{c}{24}} q^{\frac{\lambda}{2}(|\Pi_l| + |\Pi_r| - |\Lambda_l| - |\Lambda_r|)} \tilde{M}(q) \sum_{\Xi} c_{\Pi \Lambda}^\Xi \text{ch}_{\Xi_l}(U_{1/2}) \cdot \text{ch}_{\Xi_r}(U_{1/2}) , \quad (3.11)$$

where $\tilde{M}(q)$ is the modified MacMahon function

$$\tilde{M}(q) = \prod_{n=2}^{\infty} \frac{1}{(1 - q^n)^{n-1}} = \prod_{s=2}^{\infty} \prod_{n=s}^{\infty} \frac{1}{1 - q^n} , \quad (3.12)$$

and $c_{\Pi \Lambda}^\Xi$ are the Clebsch-Gordan coefficients of $\mathfrak{su}(N)$. Furthermore, we define

$$\text{ch}_\Lambda(U_h) = \sum_{T \in \text{Tab}_\Lambda} \prod_{j \in T} q^{h+j} , \quad (3.13)$$

where the matrix U_h has only the diagonal matrix elements $(U_h)_{jj} = q^{h+j}$, and the sum is over a filling of the boxes of a semistandard Young tableau of shape Λ with integers $j \geq 0$. Thus the extended vacuum character χ_0 equals

$$\begin{aligned} \chi_0(q) &= \sum_{\Pi \in \Omega} b_{\Pi; 0}^\lambda(q) = q^{-\frac{c}{24}} \tilde{M}(q) \sum_{\Xi} q^{\frac{|\Xi|}{2}} \text{ch}_\Xi(U_{\frac{1}{2}}) \cdot \text{ch}_{\Xi^t}(U_{\frac{1}{2}}) \\ &= q^{-\frac{c}{24}} \tilde{M}(q) \sum_{\Xi} \text{ch}_\Xi(U_{\frac{3}{4}}) \cdot \text{ch}_{\Xi^t}(U_{\frac{3}{4}}) , \end{aligned} \quad (3.14)$$

where Ξ runs over all representations with finitely many boxes. Finally, using (3.13) as well as the dual Cauchy identity, see e.g. [38, p. 65], this simplifies to

$$\chi_0(q) = q^{-\frac{c}{24}} \tilde{M}(q) \prod_{r, u=0}^{\infty} (1 + q^{r+u+\frac{3}{2}}) = q^{-\frac{c}{24}} \tilde{M}(q) \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} (1 + q^{n+\frac{1}{2}}) . \quad (3.15)$$

In particular, the spin spectrum of the corresponding $\mathcal{N} = 1$ \mathcal{W} -algebra consists of the bosonic currents of spin $s = 2, 3, \dots$ — this is the contribution from the MacMahon function — as well as fermionic currents of spin $s = \frac{3}{2}, \frac{5}{2}, \dots$, each again with multiplicity one. Thus the total spin spectrum consists of the minimal $\mathcal{N} = 1$ spin content, namely of the $\mathcal{N} = 1$ multiplets of spin

$$(\frac{3}{2}, 2) , \quad (\frac{5}{2}, 3) , \quad (\frac{7}{2}, 4) , \quad \dots . \quad (3.16)$$

For general Λ the analysis becomes a little more involved since the Clebsch-Gordan coefficients are no longer trivial. If we denote by χ_Λ the character of the extended representation \mathcal{H}_Λ of (3.9), we find

$$\begin{aligned}\chi_\Lambda(q) &= \sum_{\Pi \in \Omega} b_{\Pi; \Lambda}^\lambda(q) \\ &= q^{-\frac{c}{24}} \tilde{M}(q) q^{-\frac{1}{4}(|\Lambda_l| + |\Lambda_r|)} \sum_{\Pi \in \Omega} q^{\frac{|\Pi_l| + |\Pi_r|}{4}} \sum_{\Xi} c_{\Pi \Lambda}^{\Xi} \text{ch}_{\Xi_l^t}(U_{\frac{1}{2}}) \cdot \text{ch}_{\Xi_r^t}(U_{\frac{1}{2}}) .\end{aligned}\quad (3.17)$$

In order to simplify these sums one can rewrite the Clebsch-Gordan coefficients of Π and Λ in terms of those involving the corresponding subdiagrams $\Pi = (\Pi_l, \Pi_r)$ and $\Lambda = (\Lambda_l, \Lambda_r)$. This is discussed in appendix A, and it leads to (see eq. (A.9))

$$\chi_{(\Lambda_l, \Lambda_r)}(q) = \chi_0(q) \cdot \sum_{(\Xi_l, \Xi_r)} c_{(\Lambda_l^t, 0)(0, \Lambda_r)}^{(\Xi_l^t, \Xi_r)} \text{sch}_{\Xi_l^t}(\mathcal{U}_{\frac{1}{4}}) \cdot \text{sch}_{\Xi_r^t}(\mathcal{U}_{\frac{1}{4}}) , \quad (3.18)$$

where the supercharacters are defined in eq. (A.5).

3.3 The full spectrum

In order to compare to the dual higher spin theory we now need to determine the full partition function in the 't Hooft limit. In the bosonic case this turned out to be somewhat subtle [16] since certain states decouple (and become null) in this limit, and therefore should not contribute to the partition function. We shall assume that a similar phenomenon takes place in the present case, and that its effect amounts to replacing the Clebsch-Gordan coefficient

$$c_{(\Lambda_l^t, 0)(0, \Lambda_r)}^{(\Xi_l^t, \Xi_r)} \longrightarrow \delta_{\Lambda_l^t}^{\Xi_l^t} \delta_{\Lambda_r}^{\Xi_r} , \quad (3.19)$$

in close analogy to what happened in [16].¹ Then the character associated to $\Lambda = (\Lambda_l, \Lambda_r)$ becomes

$$\chi_{(\Lambda_l, \Lambda_r)}^{\text{dec}}(q) = \chi_0(q) \cdot \text{sch}_{\Lambda_l^t}(\mathcal{U}_{1/4}) \cdot \text{sch}_{\Lambda_r^t}(\mathcal{U}_{1/4}) , \quad (3.20)$$

and the full partition function equals

$$\begin{aligned}\mathcal{Z}_{\text{CFT}}^{\text{dec}} &= \sum_{\Lambda_l, \Lambda_r} |\chi_{(\Lambda_l, \Lambda_r)}^{\text{dec}}|^2 \\ &= |\chi_0|^2 \cdot \sum_{\Lambda_l, \Lambda_r} |\text{sch}_{\Lambda_l^t}(\mathcal{U}_{1/4}) \cdot \text{sch}_{\Lambda_r^t}(\mathcal{U}_{1/4})|^2 \\ &= (q\bar{q})^{-\frac{c}{24}} |\tilde{M}(q)|^2 \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} |1 + q^{n+\frac{1}{2}}|^2 \sum_{\Lambda_l, \Lambda_r} |\text{sch}_{\Lambda_l^t}(\mathcal{U}_{1/4}) \cdot \text{sch}_{\Lambda_r^t}(\mathcal{U}_{1/4})|^2 .\end{aligned}\quad (3.21)$$

¹Indeed, eq. (3.19) simply means that no boxes are allowed to cancel against anti-boxes. Without this prescription, the partition function diverges in the 't Hooft limit.

4 The $\mathcal{N} = 1$ $s\mathcal{W}_\infty$ algebra

Before we proceed to identify the dual higher spin theory, we first want to understand in more detail the most general $\mathcal{N} = 1$ superconformal $s\mathcal{W}_\infty$ algebra whose spin content agrees with (3.16). As we shall see, for each value of the central charge c , there is a unique such algebra, and it contains indeed the bosonic $\mathcal{W}_\infty[\frac{1}{2}]$ algebra as a subalgebra.

4.1 Structure of $\mathcal{N} = 1$ primaries

The analysis of the $\mathcal{N} = 1$ superconformal $s\mathcal{W}$ -algebra is most easily performed using $\mathcal{N} = 1$ superfields. In particular, the energy-momentum tensor T and the supercurrent G can be combined into the superfield $\mathbb{T} = \frac{1}{2}G + \theta T$, whose OPE is of the form

$$\mathbb{T}(Z_1)\mathbb{T}(Z_2) = \frac{c/6}{Z_{12}^3} + \frac{\frac{3}{2}\theta_{12}\mathbb{T}(Z_2)}{Z_{12}^2} + \frac{\frac{1}{2}D\mathbb{T}(Z_2)}{Z_{12}} + \frac{\theta_{12}\mathbb{T}'(Z_2)}{Z_{12}^2} + \dots, \quad (4.1)$$

where $Z_{12} = z_1 - z_2 - \theta_1\theta_2$, $\theta_{12} = \theta_1 - \theta_2$ and $D = \partial_\theta + \theta\partial$ with $D^2 = \partial^2$. A superprimary field $\mathbb{V}^{(h)} = V^{(h)} + \theta V^{h+\frac{1}{2}}$ is similarly defined by the OPE

$$\mathbb{T}(Z_1)\mathbb{V}^{(h)}(Z_2) = \frac{h\theta_{12}\mathbb{V}^{(h)}(Z_2)}{Z_{12}^2} + \frac{\frac{1}{2}D\mathbb{V}^{(h)}(Z_2)}{Z_{12}} + \frac{\theta_{12}\partial\mathbb{V}^{(h)}(Z_2)}{Z_{12}} + \dots. \quad (4.2)$$

In particular, this implies that $V^{(h)}$ and $V^{(h+\frac{1}{2})}$ are Virasoro primaries. The super OPE between two superprimary fields can be expanded in superconformal families that are obtained by acting with the negative modes of T and G . The detailed structure can be fixed by imposing associativity with the OPE of \mathbb{T} ; alternatively, one may use the general results of [39]. Our conventions for the structure constants of super OPEs follow [40], where useful selection rules have been derived.

4.2 Enumerating superprimaries

The other important ingredient for the analysis of the $s\mathcal{W}_\infty$ algebra is the structure of the various $\mathcal{N} = 1$ superprimaries that are contained in the vacuum representation. Their numbers can be easily determined using character techniques. To this end we expand the vacuum character of $s\mathcal{W}_\infty$ in terms of $\mathcal{N} = 1$ superconformal characters as²

$$\chi_0(q) = \prod_{n=2}^{\infty} \frac{1+q^{n-1/2}}{1-q^n} + \sum_h d_h q^h \prod_{n=1}^{\infty} \frac{1}{1-q^n} \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}}), \quad (4.3)$$

where the first term describes the contribution of the $\mathcal{N} = 1$ superconformal descendants of the vacuum, while d_h is the multiplicity of the $\mathcal{N} = 1$ superconformal primary of conformal dimension h . Given the explicit formula for the $\mathcal{N} = 1$ $s\mathcal{W}_\infty$ vacuum representation, see eq. (3.15), the generating function of these multiplicities turns out to equal

$$\sum_h d_h q^h = q^{5/2} + q^{7/2} + q^{9/2} + 2q^{11/2} + 2q^6 + 2q^{13/2} + 2q^7 + 5q^{15/2} + \dots. \quad (4.4)$$

²In this section we routinely drop the factor $q^{-c/24}$ from characters.

Apart from the algebra generators that appear at every half-integer conformal dimension with multiplicity one, we therefore have additional composite superprimaries, the first of which has conformal dimension $\frac{11}{2}$. We shall use the convention that the former (i.e. the algebra generators) are denoted by $\mathbb{V}^{(h)}$, while the latter will be labelled as $\mathbb{V}^{(h),a}, \mathbb{V}^{(h),b}, \dots$

4.3 Jacobi identities

With these preparations we can now discuss the actual construction of $\mathcal{N} = 1$ $s\mathcal{W}_\infty$ by imposing recursively the Jacobi identities that encode the associativity of the operator algebra; this can be done using the same techniques as in [41, 42]. As will become clear, one can in principle push the analysis to arbitrary order, but obviously the problem becomes more and more complex.

Let us begin with the ansatz for the OPE

$$\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} = \mathbb{I} + \mathbb{V}^{(\frac{7}{2})} . \quad (4.5)$$

Here we have chosen a particular normalisation for both $\mathbb{V}^{(\frac{5}{2})}$ and $\mathbb{V}^{(\frac{7}{2})}$.³ By dimension counting, also the superprimary $\mathbb{V}^{(\frac{9}{2})}$ would have been allowed to appear in this OPE, but it is forbidden by the 3-point function selection rules of [40]. The next OPE is

$$\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{7}{2})} = \mathbb{D}_{\frac{5}{2}}^{\frac{5}{2} \frac{7}{2}} \mathbb{V}^{(\frac{5}{2})} + \mathbb{V}^{(\frac{9}{2})} , \quad (4.6)$$

where the coefficient with which $\mathbb{V}^{(\frac{5}{2})}$ appears is a coupling constant. In principle also the two superprimaries with $h = \frac{11}{2}$ could have appeared in this OPE, but the associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})}$ requires that they do not. In fact, we have found empirically that the associativity constraints imply that the OPEs always respect the symmetry

$$\mathbb{V}^{(s)} \mapsto (-1)^{s+\frac{1}{2}} \mathbb{V}^{(s)} . \quad (4.7)$$

This is the natural generalisation of the $W^{(s)} \mapsto (-1)^s W^{(s)}$ automorphism symmetry of the bosonic $\mathcal{W}_\infty[\frac{1}{2}]$ algebra.

The associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})}$ not only leads to the above selection rules, but it also fixes the coupling constant to equal

$$\mathbb{D}_{\frac{5}{2}}^{\frac{5}{2} \frac{7}{2}} = \frac{192(2c+5)(7c-10)}{c(4c+21)(10c-7)} . \quad (4.8)$$

The OPEs $\frac{7}{2} \times \frac{7}{2}$ and $\frac{5}{2} \times \frac{9}{2}$:

Using the previous results, we can now build explicit expressions for the composite superprimaries

$$\mathbb{V}^{(\frac{11}{2}),a}, \quad \mathbb{V}^{(6),a}, \quad \mathbb{V}^{(6),b} . \quad (4.9)$$

³Note that these choices implicitly assume that both $\mathbb{V}^{(\frac{5}{2})}$ and $\mathbb{V}^{(\frac{7}{2})}$ actually appear in the algebra, as will be generically the case. However, when we discuss truncations of the $s\mathcal{W}_\infty$ algebra to finitely generated algebras later on, we have to be careful about choices of this kind.

Schematically, they are of the form

$$V^{(\frac{11}{2}),a} = (V^{(\frac{5}{2})} V^{(3)}) + \dots, \quad V^{(6),a} = -\frac{1}{10} (V^{(3)} V^{(3)}) + \dots, \quad V^{(6),b} = (V^{(\frac{5}{2})} V^{(\frac{7}{2})}) + \dots, \quad (4.10)$$

where (AB) denotes the normal ordered product, and the dots stand for terms involving G - and T -descendants. Then we can make the most general ansatz for the next two OPEs as

$$\begin{aligned} \mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{7}{2})} &= \mathbb{D}_0^{\frac{7}{2}\frac{7}{2}} \mathbb{I} + \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2}\frac{7}{2}} \mathbb{V}^{(\frac{7}{2})} + \mathbb{V}^{(\frac{11}{2})} + \mathbb{D}_{\frac{11}{2},a}^{\frac{7}{2}\frac{7}{2}} \mathbb{V}^{(\frac{11}{2}),a} + \mathbb{D}_{6,a}^{\frac{7}{2}\frac{7}{2}} \mathbb{V}^{(6),a} + \mathbb{D}_{6,b}^{\frac{7}{2}\frac{7}{2}} \mathbb{V}^{(6),b}, \\ \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{9}{2})} &= \mathbb{D}_{\frac{7}{2}}^{\frac{5}{2}\frac{9}{2}} \mathbb{V}^{(\frac{7}{2})} + \mathbb{D}_{\frac{11}{2}}^{\frac{5}{2}\frac{9}{2}} \mathbb{V}^{(\frac{11}{2})} + \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2}\frac{9}{2}} \mathbb{V}^{(\frac{11}{2}),a} + \mathbb{D}_{6,a}^{\frac{5}{2}\frac{9}{2}} \mathbb{V}^{(6),a} + \mathbb{D}_{6,b}^{\frac{5}{2}\frac{9}{2}} \mathbb{V}^{(6),b}. \end{aligned} \quad (4.11)$$

Again, the superprimaries with $h = \frac{13}{2}$ could have appeared, but we have found that they do not, in agreement with (4.7). Imposing the associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{7}{2})}$ then leads to the constraints

$$\begin{aligned} \mathbb{D}_0^{\frac{7}{2}\frac{7}{2}} &= \frac{192(2c+5)(7c-10)}{c(4c+21)(10c-7)}, & \mathbb{D}_{\frac{7}{2}}^{\frac{5}{2}\frac{9}{2}} &= \frac{150(4c+21)(6c-13)}{c(2c+37)(10c-7)}, \\ \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2}\frac{7}{2}} &= \frac{720(8c^2-9c-34)}{c(4c+21)(10c-7)}, & \mathbb{D}_{\frac{11}{2}}^{\frac{5}{2}\frac{9}{2}} &= \frac{5}{4}, \\ \mathbb{D}_{\frac{11}{2},a}^{\frac{7}{2}\frac{7}{2}} &= \frac{4}{5} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2}\frac{9}{2}} + \frac{46080(7c-10)}{11c(2c+37)(10c-7)}, & \mathbb{D}_{6,a}^{\frac{5}{2}\frac{9}{2}} &= -\frac{7200(14c+11)}{11c(2c+37)(10c-7)}, \\ \mathbb{D}_{6,a}^{\frac{7}{2}\frac{7}{2}} &= -\frac{11520}{11c(10c-7)}, & \mathbb{D}_{6,b}^{\frac{5}{2}\frac{9}{2}} &= 0, \\ \mathbb{D}_{6,b}^{\frac{7}{2}\frac{7}{2}} &= 0. \end{aligned} \quad (4.12)$$

The OPEs $\frac{7}{2} \times \frac{9}{2}$ and $\frac{5}{2} \times \frac{11}{2}$:

The ansatz for the next OPEs are

$$\begin{aligned} \mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{9}{2})} &= \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{5}{2})} + \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{9}{2})} + \mathbb{D}_{\frac{11}{2}}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{11}{2})} + \mathbb{D}_{\frac{11}{2},a}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{11}{2}),a} + \mathbb{D}_{6,a}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(6),a} \\ &+ \mathbb{D}_{6,b}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(6),b} + \mathbb{V}^{(\frac{13}{2})} + \mathbb{D}_{\frac{13}{2},a}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{13}{2}),a} + \mathbb{D}_{7,a}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(7),a} + \mathbb{D}_{7,b}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(7),b} \\ &+ \mathbb{D}_{\frac{15}{2}}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{15}{2})} + \mathbb{D}_{\frac{15}{2},a}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{15}{2}),a} + \mathbb{D}_{\frac{15}{2},b}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{15}{2}),b} + \mathbb{D}_{\frac{15}{2},c}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{15}{2}),c} + \mathbb{D}_{\frac{15}{2},d}^{\frac{7}{2}\frac{9}{2}} \mathbb{V}^{(\frac{15}{2}),d}, \end{aligned} \quad (4.13)$$

$$\begin{aligned} \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{11}{2})} &= \mathbb{D}_{\frac{7}{2}}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{5}{2})} + \mathbb{D}_{\frac{7}{2}}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{7}{2})} + \mathbb{D}_{\frac{9}{2}}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{9}{2})} + \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{11}{2}),a} + \mathbb{D}_{6,a}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(6),a} \\ &+ \mathbb{D}_{6,b}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(6),b} + \mathbb{D}_{\frac{13}{2}}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{13}{2})} + \mathbb{D}_{\frac{13}{2},a}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{13}{2}),a} + \mathbb{D}_{7,a}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(7),a} + \mathbb{D}_{7,b}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(7),b} \\ &+ \mathbb{D}_{\frac{15}{2}}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{15}{2})} + \mathbb{D}_{\frac{15}{2},a}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{15}{2}),a} + \mathbb{D}_{\frac{15}{2},b}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{15}{2}),b} + \mathbb{D}_{\frac{15}{2},c}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{15}{2}),c} + \mathbb{D}_{\frac{15}{2},d}^{\frac{5}{2}\frac{11}{2}} \mathbb{V}^{(\frac{15}{2}),d}. \end{aligned} \quad (4.14)$$

Note that we have to include superprimaries of conformal dimension $h = \frac{15}{2}$ since (4.7) only predicts that the elementary superprimary of that conformal dimension does not arise; indeed, it turns out that one of the composite superprimaries does indeed appear. Imposing the associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{7}{2})}$, the above coupling constants are determined, see Appendix B.1 and B.2, except for $\mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}}$ and $\mathbb{D}_{\frac{13}{2},a}^{\frac{5}{2},\frac{11}{2}}$. However, these two couplings are not actually free parameters, but just reflect the freedom that we can redefine superprimaries of the same dimension, see [42] for a similar phenomenon. For the following we shall choose the convention that

$$\mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}} = \mathbb{D}_{\frac{13}{2},a}^{\frac{5}{2},\frac{11}{2}} = 0 , \quad (4.15)$$

where the top component of $\mathbb{V}^{(\frac{13}{2}),a}$ is of the form

$$V^{(\frac{13}{2}),a} = (V^{(\frac{5}{2})} V^{(4)}) + \frac{7}{5} (V^{(3)} V^{(\frac{7}{2})}) + \dots . \quad (4.16)$$

We have checked that the associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{9}{2})}$ is then also satisfied by these OPEs.

We have also analysed the Jacobi identities involving the OPEs of

$$\mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{11}{2})} , \quad \mathbb{V}^{(\frac{9}{2})} \times \mathbb{V}^{(\frac{9}{2})} , \quad \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{13}{2})} , \quad (4.17)$$

and determined the relevant structure constants. Some of the details of this analysis are given in Appendices B.3 and B.4.

In summary, these considerations therefore suggest that the algebra does not have any free parameter beyond the central charge. In the following three subsections we shall subject these results to some independent consistency checks. First, in Section 4.4, we shall show that the algebra truncates to the finitely generated $\mathcal{N} = 1$ algebras for the appropriate values of the central charge. In the remaining two subsections we shall then demonstrate that it contains indeed the bosonic $\mathcal{W}_\infty[\frac{1}{2}]$ algebra as a subalgebra, first directly in terms of the structure constants, and then by studying the so-called minimal representations that will also play a role for the higher spin holography.

4.4 Truncation properties

We have seen in Section 3.2 that the vacuum character of the coset reproduces the spectrum of $s\mathcal{W}_\infty$ in the $N \rightarrow \infty$ limit. It is therefore natural to expect that when the central charge equals one of the coset values

$$c_N = \frac{(3N+1)(N-1)}{2(2N+1)} \quad (4.18)$$

the $s\mathcal{W}_\infty$ algebra truncates to the $\mathcal{N} = 1$ extended coset algebra, which we denote by $s\mathcal{W}_N$. This is an exceptional \mathcal{W} -algebra with spin content⁴

$$(\frac{3}{2}, 2) , \quad (\frac{5}{2}, 3) , \quad \dots , \quad (N - \frac{1}{2}, N) ,$$

⁴Let us mention that the direct construction of these algebras is complicated by the fact that the Jacobi identities of the generators can only be satisfied modulo non-trivial null fields.

see [36]. With the explicit values of the structure constants of $s\mathcal{W}_\infty$ at hand, let us check whether the latter truncates to $s\mathcal{W}_N$ at $c = c_N$. In order for this to happen, it is necessary that

$$\mathbb{D}_{s_2}^{s s_1} = 0, \quad \forall s_1 \geq N + \frac{1}{2}, \quad s_2 \leq N - \frac{1}{2}, \quad (4.19)$$

irrespective of the value of s . Similarly, for any composite field $\mathbb{V}^{(s_2),a}$ which is not part of the ideal, we must require that

$$\mathbb{D}_{s_2,a}^{s s_1} = 0, \quad (4.20)$$

provided that $s_1 \geq N + \frac{1}{2}$. Let us now consider individually the cases $N = 3, 4, 5, 6$.⁵ Demanding (4.19) for $N = 3$ requires in particular that

$$\mathbb{D}_0^{\frac{7}{2} \frac{7}{2}} = \mathbb{D}_0^{\frac{9}{2} \frac{9}{2}} = \mathbb{D}_{\frac{5}{2}}^{\frac{5}{2} \frac{7}{2}} = \mathbb{D}_{\frac{5}{2}}^{\frac{5}{2} \frac{11}{2}} = 0, \quad (4.21)$$

which leads to $c = -\frac{5}{2}$ or $c = c_3 = \frac{10}{7}$, in perfect agreement with [43]. For the $N = 4$ truncation we need to set

$$\mathbb{D}_0^{\frac{9}{2} \frac{9}{2}} = \mathbb{D}_{\frac{7}{2}}^{\frac{5}{2} \frac{9}{2}} = \mathbb{D}_{\frac{5}{2}}^{\frac{5}{2} \frac{11}{2}} = \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2} \frac{11}{2}} = \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2} \frac{9}{2}} = \mathbb{D}_{\frac{13}{2},a}^{\frac{5}{2} \frac{11}{2}} = 0,$$

which gives, using (B.16) and (B.28), $c = c_4 = \frac{13}{6}$. In order to deal with $N \geq 5$ we need to relax eq. (4.15) and fix the values of the respective structure constants from the truncation analysis. We have found that a necessary condition for the truncation to $N = 5$ to occur is that c equals either

$$c = -\frac{2}{5}, \quad \text{or} \quad c = -\frac{16}{5}, \quad \text{or} \quad c = \frac{32}{11} \equiv c_5. \quad (4.22)$$

Finally, we have checked that for $N = 6$, $c = c_6$ is a solution of the truncation constraints (but we have not determined sufficiently many structure constants in order to rule out a number of other solutions).

4.5 Bosonic subalgebra

The $\mathcal{N} = 1$ extension of the cosets (2.1) with $k = N$ implies that the chiral algebra of the bosonic coset must be a subalgebra of its extension, i.e. $\mathcal{W}_N \subset s\mathcal{W}_N$. In this section we give strong evidence for the claim that the same subalgebra structure lifts to the infinitely generated algebras

$$\mathcal{W}_\infty[\frac{1}{2}] \subset s\mathcal{W}_\infty. \quad (4.23)$$

To set up the notation, recall that the $\mathcal{W}_\infty[\mu]$ algebra is generated by primary fields $W^{(s)}$ with dimension $s = 3, 4, 5, \dots$, that we normalise as $W^{(s)} \times W^{(s)} = \frac{c}{s} \mathbb{I} + \dots$. The first few OPEs of $\mathcal{W}_\infty[\mu]$ are

$$W^{(3)} \times W^{(3)} = \frac{c}{3} \mathbb{I} + C_{33}^4 W^{(4)}, \quad (4.24)$$

⁵The case $N = 2$ is a bit unnatural since the $\mathcal{N} = 1$ Virasoro algebra is always a consistent subalgebra of $s\mathcal{W}_\infty$, irrespective of any such truncation. However, if one formally applies the above conditions for $N = 2$ one finds the two values $c = 0$ and $c = c_2 = \frac{7}{10}$.

$$W^{(3)} \times W^{(4)} = C_{34}^3 W^{(3)} + C_{34}^5 W^{(5)} , \quad (4.25)$$

where (see [15, 44])

$$(C_{33}^4)^2 = \frac{64(c+2)(\mu-3)(c(\mu+3)+2(4\mu+3)(\mu-1))}{(5c+22)(\mu-2)(c(\mu+2)+(3\mu+2)(\mu-1))} , \quad (4.26)$$

$$C_{34}^3 = \frac{3}{4} C_{44}^3 , \quad (4.27)$$

$$C_{33}^4 C_{44}^4 = \frac{48(c^2(\mu^2-19)+3c(6\mu^3-25\mu^2+15)+2(\mu-1)(6\mu^2-41\mu-41))}{(\mu-2)(5c+22)(c(\mu+2)+(3\mu+2)(\mu-1))} , \quad (4.28)$$

$$(C_{34}^5)^2 = \frac{25(5c+22)(\mu-4)(c(\mu+4)+3(5\mu+4)(\mu-1))}{(7c+114)(\mu-2)(c(\mu+2)+(3\mu+2)(\mu-1))} . \quad (4.29)$$

These expressions reduce, for $\mu = \frac{1}{2}$, to

$$(C_{33}^4)^2 = \frac{640(c+2)(7c-10)}{3(5c+22)(10c-7)} , \quad (4.30)$$

$$C_{33}^4 C_{44}^4 = \frac{96(25c^2-38c-80)}{(5c+22)(10c-7)} , \quad (4.31)$$

$$(C_{34}^5)^2 = \frac{175(5c+22)(6c-13)}{(7c+114)(10c-7)} . \quad (4.32)$$

Since the $s\mathcal{W}_\infty$ algebra contains only a single Virasoro primary field of conformal dimension 3, we must have the identification

$$W^{(3)} = \sqrt{\frac{c}{3}} V^{(3)} , \quad (4.33)$$

where the prefactor is fixed by the normalisation conventions. Evaluating the $W^{(3)}W^{(3)}$ OPE and decomposing it in terms of conformal families of primary fields, we find

$$W^{(3)} \times W^{(3)} = \frac{c}{3} \mathbb{I} + \widetilde{W}^{(4)} , \quad (4.34)$$

where $\widetilde{W}^{(4)}$ is the Virasoro primary field

$$\begin{aligned} \widetilde{W}^{(4)} = & \frac{8(7c-10)(G'G)}{(4c+21)(10c-7)} - \frac{136(7c-10)(TT)}{(4c+21)(5c+22)(10c-7)} \\ & - \frac{12(c+1)(7c-10)T''}{(4c+21)(5c+22)(10c-7)} + \frac{cV^{(4)}}{3} . \end{aligned} \quad (4.35)$$

Given the normalisation conventions of $\mathcal{W}_\infty[\frac{1}{2}]$, the correctly normalised spin 4 field is then

$$W^{(4)} = \sqrt{\frac{3(5c+22)(10c-7)}{640(c+2)(7c-10)}} \widetilde{W}^{(4)} , \quad (4.36)$$

in agreement with (4.30). Next, we can evaluate the OPE $W^{(3)}W^{(4)}$ and find that it takes the form

$$W^{(3)} \times W^{(4)} = C_{34}^3 W^{(3)} + \widetilde{W}^{(5)} , \quad (4.37)$$

where the value of C_{34}^3 is indeed in agreement with (4.27). Note that no Virasoro primary field of spin $s = 6$ appears in this OPE, again as expected from $\mathcal{W}_\infty[\frac{1}{2}]$. The field $\widetilde{W}^{(5)}$ is explicitly given as

$$\begin{aligned} \widetilde{W}^{(5)} = & \frac{\sqrt{\frac{c(5c+22)(10c-7)}{10(c+2)(7c-10)}}}{24(2c+37)(7c+114)(10c-7)} \left[-360(6c-13)(7c+114)(GV^{(\frac{5}{2})}') \right. \\ & + 600(6c-13)(7c+114)(G'V^{(\frac{5}{2})}) + 360(c+34)(6c-13)V^{(3)''} \\ & \left. - 29760(6c-13)(TV^{(3)}) + c(2c+37)(7c+114)(10c-7)V^{(5)} \right], \end{aligned} \quad (4.38)$$

where $V^{(5)}$ is the bosonic component of $\mathbb{V}^{(\frac{9}{2})}$. Again, all the OPEs required to normalise this field are available, and we find

$$W^{(5)} = \sqrt{\frac{(7c+114)(10c-7)}{175(5c+22)(6c-13)}} \widetilde{W}^{(5)}, \quad (4.39)$$

in agreement with (4.32). Finally, we can evaluate C_{44}^4 in the $W^{(4)}W^{(4)}$ OPE

$$W^{(4)} \times W^{(4)} = \frac{c}{4} \mathbb{I} + C_{44}^4 W^{(4)} + \dots, \quad (4.40)$$

and compute

$$C_{44}^4 = \frac{6\sqrt{\frac{6}{5}}(c(25c-38)-80)}{\sqrt{(c+2)(5c+22)(7c-10)(10c-7)}}, \quad (4.41)$$

again in agreement with (4.31). These checks therefore provide very convincing evidence that $s\mathcal{W}_\infty$ contains indeed $\mathcal{W}_\infty[\frac{1}{2}]$ as a subalgebra.

4.6 Minimal representations

As an independent consistency check of the analysis of the previous subsection, we can also determine the structure of the so-called minimal representations. They are characterised by the property that their character equals

$$\chi_{\min}(q) = q^h \frac{1+q^{\frac{1}{2}}}{1-q} \cdot \chi_0(q). \quad (4.42)$$

In particular, this means that the representation is highly degenerate, e.g. the only descendant at conformal dimension $h + \frac{1}{2}$ is the $G_{-1/2}$ -descendant, the only descendant at conformal dimension $h + 1$ the L_{-1} -descendant, etc. The minimal representations play an important role in the holographic duality since they correspond to the smallest matter multiplet.

In terms of OPEs, the representation generated from the super-Virasoro primary $\mathbb{P}^{(h)}$ defines a minimal representation provided that its OPEs are of the form

$$\mathbb{V}^{(s)} \times \mathbb{P}^{(h)} = \sum_{\mathbb{P}' \in \mathcal{P}} \mathbb{D}_{\mathbb{P}'}^{sh} \mathbb{P}', \quad (4.43)$$

where the superprimaries appearing in \mathcal{P} are built out of normal ordered products of $\mathbb{V}^{(s)}$, $\mathbb{P}^{(h)}$ and their derivatives, that are *linear* in $\mathbb{P}^{(h)}$ (including $\mathbb{P}^{(h)}$ itself). Indeed, this is equivalent to requiring that all the fields $\{V_{-m}^{(s)} P^{(h)} \mid m < s\}$ that appear in the singular part of the OPE (4.43) can be expressed in terms of L_{-1} and $G_{-1/2}$ descendants of $P^{(h)}$, or of states that are obtained from $P^{(h)}$ by the action of non-wedge modes, $V_{-n}^{(t)}$ with $n \geq t$; this in turn is equivalent to the characterisation of the minimal representations given in (4.42).

To determine the conformal weight of the minimal representation we can proceed as in [41, 42]: we make the most general ansatz of the form (4.43), and check that it is associative with respect to further OPEs with $\mathbb{V}^{(s')}$. In order to determine which fields may appear in (4.43), we count the super-Virasoro primaries that appear in the representation generated from $\mathbb{P}^{(h)}$; using similar arguments as around eq. (4.4), they are counted by the generating function

$$\sum_{h'} \hat{d}_{h'} q^{h'} = q^h \left[1 + q^{5/2} + q^3 + 2q^{7/2} + 2q^4 + 3q^{9/2} + 3q^5 + \dots \right]. \quad (4.44)$$

The first few OPEs are rather simple. For example, for $s = \frac{5}{2}$ we have simply

$$\mathbb{V}^{(\frac{5}{2})} \times \mathbb{P}^{(h)} = \mathbb{D}_{\frac{5}{2}h}^{\frac{5}{2}} \mathbb{P}^{(h)}, \quad (4.45)$$

since no composite superprimary can occur. For $s = \frac{7}{2}$ we find on the other hand

$$\mathbb{V}^{(\frac{7}{2})} \times \mathbb{P}^{(h)} = \mathbb{D}_{\frac{7}{2}h}^{\frac{7}{2}} \mathbb{P}^{(h)} + \mathbb{D}_{h+\frac{5}{2}}^{\frac{7}{2}} \mathbb{P}^{(h+\frac{5}{2})} + \mathbb{D}_{h+3}^{\frac{7}{2}} \mathbb{P}^{(h+3)}, \quad (4.46)$$

where $\mathbb{P}^{(h+\frac{5}{2})}$ and $\mathbb{P}^{(h+3)}$ are the composite superprimaries of conformal dimension $h+\frac{5}{2}$ and $h+3$, respectively. (It is not hard to write down explicit expressions for these composite fields, but we shall refrain from doing so here.) Then we can impose the associativity $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} \times \mathbb{P}^{(h)}$, from which it follows that the conformal dimension of $\mathbb{P}^{(h)}$ must satisfy

$$2h(h+2) + c(4h-1) = 0. \quad (4.47)$$

Furthermore, the leading couplings (from which one can read off the eigenvalues of the respective zero modes) equal

$$\left(\mathbb{D}_{\frac{5}{2}h}^{\frac{5}{2}} \right)^2 = -\frac{(h+1)^2(4h-1)(10h-1)}{h(h+2)(2h+7)}, \quad (4.48)$$

$$\mathbb{D}_{\frac{7}{2}h}^{\frac{7}{2}} = -\frac{12(2h-5)(2h-1)(2h+3)(4h-1)(7h-1)}{h(h+2)(2h+7)(8h^2-68h+21)}. \quad (4.49)$$

The equation for the conformal dimension (4.47) has two solutions, namely

$$h_{\pm} = -(1+c) \pm \frac{1}{2} \sqrt{(c+2)(c+\frac{1}{2})}, \quad (4.50)$$

which agree precisely with the two solutions given in eq. (3.8) of [15] for $\mu = \frac{1}{2}$. For $c = c_N$, see eq. (4.18), they simplify to

$$h_+ = \frac{N-1}{2(2N+1)} \equiv h(0; \text{f}), \quad \text{and} \quad h_- = -\frac{3N+1}{2}. \quad (4.51)$$

The first solution agrees therefore with the $(0; \mathfrak{f})$ solution of the bosonic \mathcal{W}_∞ algebra at $\mu = \frac{1}{2}$. This ties in nicely with the fact that it follows from eq. (3.18) that⁶

$$\chi_{(\mathfrak{f},0)}(q) = \chi_0(q) \cdot \text{sch}_{\square}(\mathcal{U}_{1/4}) = \chi_0(q) \cdot \frac{q^{\frac{1}{4}}(1+q^{\frac{1}{2}})}{(1-q)} \quad (4.52)$$

defines indeed a minimal representation in the 't Hooft limit. (Note that in the 't Hooft limit, $h_+ = \frac{1}{4}$.) These results provide an independent check of our claim that $\mathcal{W}_\infty[\frac{1}{2}]$ is a subalgebra of $s\mathcal{W}_\infty$ (for all values of the central charge).

5 The dual higher spin point of view

We now finally turn to the description of the dual higher spin gravity theory. We begin by describing the underlying higher spin algebra.

5.1 The higher spin algebra

A higher spin algebra corresponding to the spectrum (3.16) was constructed some time ago in [33]. This algebra may be described as a certain restriction of the $\mathcal{N} = 2$ shs $[\mu]$ algebra at the special point $\mu = \frac{1}{2}$. Recall that the $\mathcal{N} = 2$ supersymmetric higher spin algebra shs $[\mu]$ is a one parameter family of Lie superalgebras [45] which can be defined as the quotient

$$\text{shs}[\mu] \oplus \mathbb{C} = \frac{U(\mathfrak{osp}(1|2))}{\langle C^{\mathfrak{osp}(1|2)} - \frac{1}{4}\mu(\mu-1)\mathbf{1} \rangle} , \quad (5.1)$$

where $U(\mathfrak{osp}(1|2))$ denotes the universal enveloping algebra of $\mathfrak{osp}(1|2)$, and $C^{\mathfrak{osp}}$ is the corresponding quadratic Casimir. In order to describe this more explicitly, let us define the oscillator algebra

$$[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu k) , \quad \{k, y_\alpha\} = 0 , \quad k^2 = \mathbf{1} , \quad (5.2)$$

where $\alpha, \beta = 1, 2$, $\frac{\nu}{2} = (\mu - \frac{1}{2})$, and $\epsilon_{12} = +1 = -\epsilon_{21}$. Then the quotient (5.1) can be realised by identifying the generators of $\mathfrak{osp}(1|2)$ with

$$G_{\frac{1}{2}} = \frac{1}{2}e^{-i\frac{\pi}{4}}y_1 , \quad G_{-\frac{1}{2}} = \frac{1}{2}e^{-i\frac{\pi}{4}}y_2 , \quad L_1 = \frac{i}{2}y_1^2 , \quad L_{-1} = \frac{i}{2}y_2^2 , \quad L_0 = \frac{i}{4}(y_1y_2 + y_2y_1) ,$$

since

$$C^{\mathfrak{osp}(1|2)} = L_0^2 - \frac{1}{2}\{L_1, L_{-1}\} + \frac{1}{4}[G_{\frac{1}{2}}, G_{-\frac{1}{2}}] = \frac{\nu^2}{4}k^2 - \frac{1}{16} . \quad (5.3)$$

The elements generating shs $[\mu]$ can thus be written as symmetric products of the oscillators y_{α_i} , $\alpha_i = 1, 2$ [45]

$$V_m^{(s)\pm} = y_{(\alpha_1 \dots \alpha_n)}(\mathbf{1} \pm k) , \quad (5.4)$$

where $V_m^{(s)}$ has ‘spin’ $s = 1 + \frac{n}{2}$ with $n \geq 0$ — for $n = 0$, $V_m^{(1)\pm} \equiv \pm k$, since the $\mathbf{1}$ generator is central and is not part of shs $[\mu]$, see (5.1). Here m takes the values $2m = N_1 - N_2$, where

⁶Note the potentially confusing notation $(0; \mathfrak{f}) \cong (\mathfrak{f}, 0)$, where the latter refers to the extended representation, see (3.9), with $\Lambda_l = \mathfrak{f}$ and $\Lambda_r = 0$.

$N_{1,2}$ is the number of $y_{1,2}$, and hence lies in the range $-s+1 \leq m \leq s-1$. For $\mu = \frac{1}{2}$, i.e. $\nu = 0$, it is consistent [45] to restrict the generators of $\text{shs}[\mu]$ to the k independent part of eq. (5.4) since k is never generated by any commutators, see eq. (5.2). This construction obviously leads to an algebra that is generated by

$$V_m^{(s)} \propto V_m^{(s)+} + V_m^{(s)-} , \quad (5.5)$$

i.e. each spin $s \geq \frac{3}{2}$ appears only once. This resulting algebra, which we shall denote as $\text{shs}(1|2)$ as in [46], is isomorphic to the symplectic higher spin algebra $\text{shs}'_\rho(1)$ of [33], as was pointed out in [45]. In [33] the strategy for the construction of this algebra was different since they used a more geometric approach; the algebra was then subsequently employed in [46] to construct a consistent action in $d=2+1$, using a Chern-Simons action based on $\text{shs}(1|2) \oplus \text{shs}(1|2)$.

For the following it will also be important to understand the simplest matter field of this theory. Because $\mathfrak{osp}(1|2)$ is a subalgebra of $\text{shs}(1|2)$ and $\text{shs}(1|2)$ is a quotient of $U(\mathfrak{osp}(1|2))$, any irreducible representation of $\mathfrak{osp}(1|2)$ for which the quadratic Casimir takes the value $C^{\mathfrak{osp}(1|2)} = -\frac{1}{16}$, see eq. (5.1), leads to an irreducible representation of $\text{shs}(1|2)$. On a highest weight state $|h\rangle$ of $\mathfrak{osp}(1|2)$ the quadratic Casimir equals

$$C^{\mathfrak{osp}(1|2)}|h\rangle = h \left(h - \frac{1}{2} \right) |h\rangle , \quad (5.6)$$

and hence $h = \frac{1}{4}$ leads to $C^{\mathfrak{osp}(1|2)} = -\frac{1}{16}$. The character of this ‘minimal’ representation is of the form

$$\chi_{\min}^{\text{shs}}(q) = \frac{q^{\frac{1}{4}}(1+q^{\frac{1}{2}})}{(1-q)} , \quad (5.7)$$

i.e. it agrees with the ‘wedge part’ of the minimal representation (4.42).

5.2 The higher spin algebra as the wedge algebra

Based on the general philosophy of [14], one should expect that the Lie superalgebra $\text{shs}(1|2)$ agrees precisely with the so-called wedge subalgebra of $s\mathcal{W}_\infty$; the latter is generated by the wedge modes $\{V_m^{(s)} \mid |m| < s\}$ in the $c \rightarrow \infty$ limit. In the following we want to confirm that this expectation is indeed borne out.

Turning the OPEs of Section 4.3 into (anti-)commutators following [47], restricting to the wedge, and finally taking the $c \rightarrow \infty$ limit, we find

$$[V_m^{(s)}, V_{m'}^{(s')}] = \sum_{s''} P_{s''}^{ss'}(m, m') d_{s''}^{ss'} V_{m+m'}^{(s'')} , \quad (5.8)$$

where we have set $V^{(\frac{3}{2})} \equiv \frac{1}{2}G$ and $V^{(2)} \equiv T$, and it is understood that the left-hand-side is a commutator or anti-commutator as appropriate. The polynomials $P_{s''}^{ss'}(m, m')$ only describe the mode dependence of the commutators and are entirely fixed by global conformal symmetry

$$P_{j''+1}^{j+1, j'+1}(m, m') = \sum_{r=0}^{j+j'-j''} \binom{j+m}{j+j'-j''-r} \frac{(-1)^r (j-j'+j''+1)_{(r)} (j''+m+m'+1)_{(r)}}{r! (2j''+2)_{(r)}} \quad (5.9)$$

where $(a)_{(n)} = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol. All the non-trivial information about the algebra is contained in the structure constants

$$d_{s''}^{ss'} = \lim_{c \rightarrow \infty} D_{s''}^{ss'} , \quad (5.10)$$

where $D_{s''}^{ss'}$ are the structure constants of the different component fields; in terms of the superprimary constants we have the relations, see [40]

$$\mathbb{D}_{s''}^{ss'} = 2s'' D_{s''+\frac{1}{2}}^{ss'} = D_{s''}^{s+\frac{1}{2} s'} = (-1)^{2s+1} D_{s''}^{s s'+\frac{1}{2}} = \frac{(-1)^{2s+1} 2s''}{s+s'+s''-\frac{1}{2}} D_{s''+\frac{1}{2}}^{s+\frac{1}{2} s'+\frac{1}{2}} . \quad (5.11)$$

Furthermore, the coupling to the energy momentum tensor multiplet $\mathbb{V}^{\frac{3}{2}}$ is determined by

$$\mathbb{D}_{\frac{3}{2}}^{ss'} = \delta_{ss'} \frac{6s}{c} \mathbb{D}_0^{ss} . \quad (5.12)$$

To the extent to which we have determined the commutation relations of $s\mathcal{W}_\infty$, we have verified that the commutation relations (5.8) reproduce those of the Lie superalgebra $\text{shs}(1|2)$ provided the $\mathfrak{osp}(1|2)$ highest weight components are identified as

$$V_{\frac{3}{2}}^{\frac{5}{2}} = -\frac{e^{\frac{3\pi i}{4}}}{4\sqrt{10}} y_1^3 , \quad V_{\frac{5}{2}}^{\frac{7}{2}} = \frac{e^{\frac{5\pi i}{4}}}{10} y_1^5 , \quad V_{\frac{7}{2}}^{\frac{9}{2}} = -\frac{e^{\frac{7\pi i}{4}}}{2\sqrt{10}} y_1^7 . \quad (5.13)$$

This provides very convincing evidence for the fact that the wedge subalgebra of $s\mathcal{W}_\infty$ is indeed isomorphic to $\text{shs}(1|2)$.

5.3 The thermal partition function of the dual higher spin theory

Finally, we want to compare the thermal partition function of the dual higher spin theory to the 't Hooft limit of the CFT partition function. First recall that the contribution of a bosonic higher spin field of spin s to the thermal 1-loop partition function on AdS equals [48]

$$Z_B^{(s)} = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2} , \quad (5.14)$$

while that of a half-integer spin s fermion is [29]

$$Z_F^{(s)} = \prod_{n=s-\frac{1}{2}}^{\infty} |1 + q^{n+\frac{1}{2}}|^2 . \quad (5.15)$$

The vacuum character of eq. (3.21) is then precisely reproduced by a tower of massless bosonic and fermionic gauge fields

$$\mathcal{Z}_{\text{gauge}} = \prod_{s=2}^{\infty} Z_B^{(s)} Z_F^{(s-\frac{1}{2})} , \quad (5.16)$$

with a spin content of $\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \dots$. This is as expected since the currents of the conformal field theory should correspond to the massless gauge fields of the higher spin theory.

In order to account for the additional contributions to the partition function, we now need to add matter fields to the higher spin theory. It was shown in [49] that a massive complex scalar field contributes

$$Z_{\text{c-scalar}}^h = \prod_{j,j'=0}^{\infty} \frac{1}{(1 - q^{h+j} \bar{q}^{h+j'})^2}, \quad (5.17)$$

where the mass equals $M^2 + 1 = (\Delta - 1)^2 = (2h - 1)^2$ in terms of the conformal dimension h of the boundary excitation. Similarly, a massive Dirac fermion whose dual conformal dimension equals $(h + \frac{1}{2}, h)$ and $(h, h + \frac{1}{2})$ leads to [29]

$$Z_{\text{spinor}}^h = \prod_{j,j'=0}^{\infty} (1 + q^{h+j} \bar{q}^{h+\frac{1}{2}+j'}) (1 + q^{h+\frac{1}{2}+j} \bar{q}^{h+j'}) , \quad (5.18)$$

where the mass squared is given by $m^2 = (\Delta_F - 1)^2 = (2h - \frac{1}{2})^2$ with $\Delta_F = 2h + \frac{1}{2}$ the total scaling dimension.

Next we observe, using the same techniques as in [30], that

$$\begin{aligned} \frac{1}{\text{sdet}(1 - \mathcal{U}_h \otimes \mathcal{U}_h^*)} &= \sum_{\Xi} \text{sch}_{\Xi}(\mathcal{U}_h) \cdot \overline{\text{sch}_{\Xi}(\mathcal{U}_h)} \\ &= \prod_{j,j'=0}^{\infty} \frac{(1 + q^{h+j} \bar{q}^{h+\frac{1}{2}+j'}) (1 + q^{h+\frac{1}{2}+j} \bar{q}^{h+j'})}{(1 - q^{h+j} \bar{q}^{h+j'}) (1 - q^{h+\frac{1}{2}+j} \bar{q}^{h+\frac{1}{2}+j'})}, \end{aligned} \quad (5.19)$$

where sdet denotes the superdeterminant. Thus, up to the contribution from the gauge fields $\mathcal{Z}_{\text{gauge}}$, the whole conformal field theory partition function, eq. (3.21), equals exactly the square of eq. (5.19) with $h = \frac{1}{4}$. In terms of matter fields, on the other hand, the square of eq. (5.19) just describes the $\mathcal{N} = 1$ matter multiplet consisting of two complex scalars of mass squared $M^2 = -\frac{3}{4}$, one with conformal dimension $(\frac{1}{4}, \frac{1}{4})$, and one with $(\frac{3}{4}, \frac{3}{4})$, as well as two massless Dirac fermions each of conformal dimension $(\frac{3}{4}, \frac{1}{4})$ and $(\frac{1}{4}, \frac{3}{4})$. Thus the higher spin theory consists of the higher spin gauge fields of spin $s = \frac{3}{2}, 2, \frac{5}{2}, 3, \dots$, together with the $\mathcal{N} = 1$ matter multiplet

$$\mathcal{Z}_{\text{matter}}^h = Z_{\text{c-scalar}}^h (Z_{\text{spinor}}^h)^2 Z_{\text{c-scalar}}^{h+\frac{1}{2}} \quad (5.20)$$

with $h = \frac{1}{4}$.

6 Conclusions

In this paper we have studied the minimal $\mathcal{N} = 1$ superconformal $s\mathcal{W}_{\infty}$ theory, and identified its higher spin dual. In particular, we have analysed the structure of the most general $s\mathcal{W}_{\infty}$ algebra that is generated by one field of each half-integer spin $s \geq \frac{3}{2}$, and we have found that it does not have any free parameter, except for the central charge. We have also shown that $s\mathcal{W}_{\infty}$ contains the bosonic $\mathcal{W}_{\infty}[\frac{1}{2}]$ algebra as a subalgebra, and indeed the $\mathcal{N} = 1$ theory can be obtained by extending the bosonic $\mathcal{W}_{\infty}[\mu]$ theory at $\mu = \frac{1}{2}$. From

that point of view, the $\mathcal{N} = 1$ theory is described by a non-diagonal modular invariant of the bosonic $\mathcal{W}_\infty[\frac{1}{2}]$ algebra.

The corresponding higher spin theory can be described in terms of a Chern-Simons theory based on the algebra $\text{shs}(1|2)$. (Note that the $\mathcal{N} = 1$ higher spin theory considered in [31] has a different spin content and is instead described by a truncation of the $\mathcal{N} = 2$ higher spin theory.) As evidence for the duality we have checked that the ‘wedge’ subalgebra of $s\mathcal{W}_\infty$ is indeed $\text{shs}(1|2)$ — this is believed to be equivalent to the statement that the asymptotic symmetry algebra of the Chern-Simons theory based on $\text{shs}(1|2)$ is $s\mathcal{W}_\infty$. We have also confirmed that the partition function of the $\mathcal{N} = 1$ minimal models is reproduced, in the ’t Hooft limit, by the thermal 1-loop partition function of the $\text{shs}(1|2)$ higher spin theory on AdS_3 , where in addition to the massless higher spin fields an $\mathcal{N} = 1$ matter multiplet has been added.

Our duality provides the first interesting example where the non-diagonal modular invariant of a \mathcal{W}_∞ theory has been identified with a dual higher spin theory. It would be very interesting to study other non-diagonal modular invariants of $\mathcal{W}_\infty[\mu]$ (maybe for special values of μ), and see whether they also have interesting higher spin bulk duals.

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A Supersymmetric form of branching functions

The Clebsch-Gordan coefficients for the mixed covariant-contravariant tensor representations of $U(N)$, which are labeled by pairs of Young diagrams, have been computed in [50] (see also [51])

$$c_{(\Pi_l, \Pi_r)(\Lambda_l, \Lambda_r)}^{(\alpha, \beta)} = \sum_{\pi, \xi, \omega, \gamma, \delta, \epsilon} c_{\pi \xi}^{\Lambda_l} c_{\omega \xi}^{\alpha} c_{\omega \gamma}^{\Pi_l} c_{\gamma \delta}^{\Lambda_r} c_{\delta \epsilon}^{\beta} c_{\pi \epsilon}^{\Pi_r}. \quad (\text{A.1})$$

This was done with the help of an older formula

$$c_{(\Lambda_l, 0)(0, \Lambda_r)}^{(\Xi_l, \Xi_r)} = \sum_{\pi} c_{\pi \Xi_l}^{\Lambda_l} c_{\pi \Xi_r}^{\Lambda_r} \quad (\text{A.2})$$

derived in [52], and the techniques developed in [53]. The complicated nature of (A.1) is due to the cancellation between boxes and antiboxes in the tensor product of $\Lambda = (\Lambda_l, \Lambda_r)$ with $\Pi = (\Pi_l, \Pi_r)$. Indeed, for $|\pi| = 0$ and $|\gamma| = 0$ no boxes have canceled against anti-boxes, and eq. (A.1) factorises to $c_{\Pi_l \Lambda_l}^{\alpha} c_{\Pi_r \Lambda_r}^{\beta}$.

To illustrate the assembling of the branching function (3.17) into the supersymmetric form (3.18), let us consider first the simpler case where Λ does not contain any anti-boxes, i.e. $\Lambda = (\Lambda_l, 0)$. Then eq. (A.1) simplifies to

$$c_{(\Pi^t, \Pi)(\Lambda_l, 0)^*}^{(\alpha, \beta)} = \sum_{\gamma, \delta} c_{\gamma \alpha}^{\Pi^t} c_{\gamma \delta}^{\Lambda_l} c_{\Pi \delta}^{\beta}, \quad (\text{A.3})$$

where we recall the conjugation operation $(\Lambda_l, \Lambda_r)^* = (\Lambda_r, \Lambda_l)$ and note that the Clebsch-Gordon coefficient $c_{\pi\xi}^0$ is only non-zero (and equal to one) if $\pi = \xi = 0$. Plugging this relation into eq. (3.17) leads to

$$\begin{aligned}
\chi_{(\Lambda_l, 0)}(q) &= q^{-\frac{c}{24}} \tilde{M}(q) q^{-\frac{|\Lambda_l|}{4}} \sum_{\Pi, \gamma, \delta, \alpha} q^{\frac{|\Pi|}{2}} c_{\gamma\delta}^{\Lambda_l} c_{\gamma\alpha}^{\Pi t} \text{ch}_{\Pi^t}(U_{\frac{1}{2}}) \cdot \text{ch}_{\delta^t}(U_{\frac{1}{2}}) \cdot \text{ch}_{\alpha^t}(U_{\frac{1}{2}}) \\
&= q^{-\frac{c}{24}} \tilde{M}(q) q^{-\frac{|\Lambda_l|}{4}} \sum_{\gamma, \delta, \alpha} c_{\gamma\delta}^{\Lambda_l} \text{ch}_{\gamma}(U_1) \cdot \text{ch}_{\delta^t}(U_{\frac{1}{2}}) \cdot \text{ch}_{\alpha}(U_1) \cdot \text{ch}_{\alpha^t}(U_{\frac{1}{2}}) \\
&= q^{-\frac{c}{24}} \tilde{M}(q) \sum_{\alpha} \text{ch}_{\alpha^t}(U_{\frac{3}{4}}) \cdot \text{ch}_{\alpha}(U_{\frac{3}{4}}) \sum_{\gamma, \delta} c_{\gamma\delta}^{\Lambda_l^t} \text{ch}_{\delta}(U_{\frac{1}{4}}) \cdot \text{ch}_{\gamma^t}(U_{\frac{3}{4}}) \\
&= \chi_0(q) \cdot \text{sch}_{\Lambda_l^t}(\mathcal{U}_{\frac{1}{4}}), \tag{A.4}
\end{aligned}$$

where we have used the invariance of the Littlewood-Richardson coefficients under transposition, together with $|\Lambda_l| = |\pi| + |\xi|$. Furthermore, we have applied in the last step eq. (3.14), as well as

$$\text{sch}_{\Lambda}(\mathcal{U}_{\frac{1}{4}}) = \sum_{\gamma, \delta} c_{\gamma\delta}^{\Lambda} \text{ch}_{\delta}(U_{\frac{1}{4}}) \cdot \text{ch}_{\gamma^t}(U_{\frac{3}{4}}) \tag{A.5}$$

that follows from [30, eq. (A.7) and (A.9)].

In order to treat the general case, we need two more character identities. The first one is a generalisation of the Cauchy identity, see [38, p. 93]

$$\sum_{\rho} \text{ch}_{\rho/\lambda^t}(U_{\frac{3}{4}}) \cdot \text{ch}_{\rho^t/\mu}(U_{\frac{3}{4}}) = \prod_{s=1}^{\infty} \prod_{n=s}^{\infty} (1 + q^{n+\frac{1}{2}}) \times \sum_{\beta} \text{ch}_{\lambda/\beta^t}(U_{\frac{3}{4}}) \cdot \text{ch}_{\mu^t/\beta}(U_{\frac{3}{4}}), \tag{A.6}$$

where $\text{ch}_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} \text{ch}_{\nu}$ are the skew Schur functions. The second identity

$$\sum_{\gamma} \text{ch}_{\lambda^t/\gamma^t}(U_{\frac{1}{4}}) \cdot \text{ch}_{\gamma/\beta}(U_{\frac{3}{4}}) = \sum_{\gamma, \epsilon, \delta} c_{\gamma^t\beta^t}^{\lambda^t} c_{\epsilon\delta^t}^{\gamma^t} \text{ch}_{\epsilon}(U_{\frac{1}{4}}) \cdot \text{ch}_{\delta}(U_{\frac{3}{4}}) = \sum_{\gamma} c_{\gamma\beta}^{\lambda} \text{sch}_{\gamma^t}(\mathcal{U}_{\frac{1}{4}}) \tag{A.7}$$

follows from the definition of the skew Schur functions, the identity (A.5), together with the associativity of the tensor product $\delta^t \otimes \beta^t \otimes \epsilon$ which implies that

$$\sum_{\gamma} c_{\gamma^t\epsilon}^{\lambda^t} c_{\beta\delta}^{\gamma} = \sum_{\gamma} c_{\gamma^t\epsilon}^{\lambda^t} c_{\beta^t\delta^t}^{\gamma^t} = \sum_{\gamma} c_{\gamma^t\beta^t}^{\lambda^t} c_{\epsilon\delta^t}^{\gamma^t}. \tag{A.8}$$

Consider now the general case. Plugging eq. (A.1) into eq. (3.17) and factorising characters we get

$$\begin{aligned}
\chi_{(\Lambda_l, \Lambda_r)} &= q^{-\frac{c}{24}} \tilde{M}(q) q^{-\frac{|\Lambda_l|+|\Lambda_r|}{4}} \sum_{\substack{\Pi, \pi, \xi, \omega \\ \gamma, \delta, \epsilon, \alpha, \beta}} q^{\frac{|\Pi|}{2}} c_{\pi\xi}^{\Lambda_r} c_{\omega\gamma}^{\Pi t} c_{\gamma\delta}^{\Lambda_l} c_{\pi\epsilon}^{\Pi} \text{ch}_{\omega^t}(U_{\frac{1}{2}}) \text{ch}_{\xi^t}(U_{\frac{1}{2}}) \text{ch}_{\delta^t}(U_{\frac{1}{2}}) \text{ch}_{\epsilon^t}(U_{\frac{1}{2}}) \\
&= q^{-\frac{c}{24}} \tilde{M}(q) q^{-\frac{|\Lambda_l|+|\Lambda_r|}{4}} \sum_{\Pi, \pi, \gamma} q^{\frac{|\Pi|}{2}} \text{ch}_{\Pi/\gamma^t}(U_{\frac{1}{2}}) \text{ch}_{\Pi^t/\pi^t}(U_{\frac{1}{2}}) \text{ch}_{\Lambda_r^t/\pi^t}(U_{\frac{1}{2}}) \text{ch}_{\Lambda_l^t/\gamma^t}(U_{\frac{1}{2}}) \\
&= q^{-\frac{c}{24}} \tilde{M}(q) \sum_{\Pi, \pi, \gamma} \text{ch}_{\Pi/\gamma^t}(U_{\frac{3}{4}}) \cdot \text{ch}_{\Pi^t/\pi^t}(U_{\frac{3}{4}}) \cdot \text{ch}_{\Lambda_r^t/\pi^t}(U_{\frac{1}{4}}) \cdot \text{ch}_{\Lambda_l^t/\gamma^t}(U_{\frac{1}{4}}).
\end{aligned}$$

Using now eq. (A.6) to perform the sum over Π we obtain

$$\begin{aligned}
\chi_{(\Lambda_l, \Lambda_r)}(q) &= \chi_0(q) \sum_{\beta, \pi, \gamma} \text{ch}_{\gamma/\beta^t}(U_{\frac{3}{4}}) \cdot \text{ch}_{\pi/\beta}(U_{\frac{3}{4}}) \cdot \text{ch}_{\Lambda_r^t/\pi^t}(U_{\frac{1}{4}}) \cdot \text{ch}_{\Lambda_l^t/\gamma^t}(U_{\frac{1}{4}}) \\
&= \chi_0(q) \sum_{\beta, \pi, \gamma} c_{\gamma\beta^t}^{\Lambda_l} c_{\beta\pi}^{\Lambda_r} \text{sch}_{\gamma^t}(\mathcal{U}_{\frac{1}{4}}) \cdot \text{sch}_{\pi^t}(\mathcal{U}_{\frac{1}{4}}) \\
&= \chi_0(q) \sum_{\gamma, \pi} c_{(\Lambda_l^t, 0)(0, \Lambda_r)}^{(\gamma^t, \pi)} \text{sch}_{\gamma^t}(\mathcal{U}_{\frac{1}{4}}) \cdot \text{sch}_{\pi^t}(\mathcal{U}_{\frac{1}{4}}) ,
\end{aligned} \tag{A.9}$$

where in the second step we have used eq. (A.7), and in the last step eq. (A.2). Note that for $\Lambda = (\Lambda_l, 0)$, i.e. $\Lambda_r = 0$, the result indeed reduces to eq. (A.4).

B Explicit constraints on the OPE coefficients from associativity

In this appendix, we give more details about the results of studying the associativity constraints. The normalisation of higher spin composites can be fixed by comparing with the leading terms listed in Appendix C.

B.1 The OPE $\frac{7}{2} \times \frac{9}{2}$

The ansatz for the OPE was given in (4.13). The associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{7}{2})}$ leads to the constraints that the couplings take the values

$$\mathbb{D}_{\frac{5}{2}}^{\frac{7}{2} \frac{9}{2}} = \frac{28800(2c+5)(6c-13)(7c-10)}{c^2(2c+37)(10c-7)^2} , \tag{B.1}$$

$$\mathbb{D}_{\frac{9}{2}}^{\frac{7}{2} \frac{9}{2}} = \frac{192(110c^3 + 881c^2 - 3439c - 8766)}{c(2c+37)(4c+21)(10c-7)} , \tag{B.2}$$

$$\mathbb{D}_{6,b}^{\frac{7}{2} \frac{9}{2}} = \frac{8400(6c-13)}{c(2c+37)(10c-7)} , \tag{B.3}$$

$$\mathbb{D}_{\frac{13}{2},a}^{\frac{7}{2} \frac{9}{2}} = \frac{35(20c+283)}{624(2c+53)} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2} \frac{9}{2}} + \frac{5}{6} \mathbb{D}_{\frac{13}{2},a}^{\frac{5}{2} \frac{11}{2}} + \frac{700(6c-13)(2345c+26774)}{143c(2c+37)(2c+53)(10c-7)} , \tag{B.4}$$

$$\mathbb{D}_{7,a}^{\frac{7}{2} \frac{9}{2}} = \frac{1200(94c+375)}{13c(2c+37)(10c-7)} , \tag{B.5}$$

$$\mathbb{D}_{\frac{15}{2},c}^{\frac{7}{2} \frac{9}{2}} = -\frac{86400(c+20)(6c-13)}{7c(2c+37)(10c-7)(13c+162)} . \tag{B.6}$$

Furthermore, the other couplings vanish

$$\mathbb{D}_{\frac{11}{2}}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{\frac{11}{2},a}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{6,a}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{7,b}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{\frac{15}{2}}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{\frac{15}{2},a}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{\frac{15}{2},b}^{\frac{7}{2} \frac{9}{2}} = \mathbb{D}_{\frac{15}{2},d}^{\frac{7}{2} \frac{9}{2}} = 0 . \tag{B.7}$$

B.2 The OPE $\frac{5}{2} \times \frac{11}{2}$

The ansatz for the OPE was given in (4.14). The associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{7}{2})}$ leads to the constraints that the couplings take the values

$$\mathbb{D}_{\frac{5}{2}}^{\frac{5}{2} \frac{11}{2}} = \frac{34560(2c+5)(6c-13)(7c-10)(17c-24)}{11c^2(2c+37)(2c+53)(10c-7)^2}$$

$$- \frac{4(2c+5)(20c^3+1441c^2+7180c+2384)}{5c(c+11)(2c+53)(10c-7)} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}}, \quad (\text{B.8})$$

$$\mathbb{D}_{\frac{9}{2},\frac{11}{2}}^{\frac{5}{2}} = \frac{576(1210c^3+17801c^2+50654c-327272)}{55c(2c+37)(2c+53)(10c-7)} - \frac{3(17c-24)}{25(2c+53)} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}}, \quad (\text{B.9})$$

$$\mathbb{D}_{6,b}^{\frac{5}{2},\frac{11}{2}} = \frac{7(4c+171)}{10(2c+53)} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}} - \frac{10080(6c-13)(23c-11)}{11c(2c+37)(2c+53)(10c-7)}, \quad (\text{B.10})$$

$$\mathbb{D}_{\frac{13}{2}}^{\frac{5}{2},\frac{11}{2}} = \frac{6}{5}, \quad (\text{B.11})$$

$$\mathbb{D}_{7,a}^{\frac{5}{2},\frac{11}{2}} = \frac{(4c-11)}{26(2c+53)} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}} + \frac{12960(334c^2+4983c+4224)}{143c(2c+37)(2c+53)(10c-7)}, \quad (\text{B.12})$$

$$\mathbb{D}_{\frac{15}{2},c}^{\frac{5}{2},\frac{11}{2}} = \frac{86400(c+20)(6c-13)(23c-11)}{77c(2c+37)(2c+53)(10c-7)(13c+162)} - \frac{6(c+20)(4c+171)}{7(2c+53)(13c+162)} \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{9}{2}}. \quad (\text{B.13})$$

Furthermore, the remaining couplings vanish

$$\mathbb{D}_{\frac{7}{2}}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{6,a}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{7,b}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{\frac{15}{2}}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{\frac{15}{2},a}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{\frac{15}{2},b}^{\frac{5}{2},\frac{11}{2}} = \mathbb{D}_{\frac{15}{2},d}^{\frac{5}{2},\frac{11}{2}} = 0. \quad (\text{B.14})$$

B.3 The OPE $\frac{7}{2} \times \frac{11}{2}$

The ansatz for the OPE $\mathbb{V}(\frac{7}{2}) \times \mathbb{V}(\frac{11}{2})$ is

$$\begin{aligned} \mathbb{V}(\frac{7}{2}) \times \mathbb{V}(\frac{11}{2}) &= \mathbb{D}_{\frac{5}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}(\frac{5}{2}) + \mathbb{D}_{\frac{7}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}(\frac{7}{2}) + \mathbb{D}_{\frac{9}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}(\frac{9}{2}) + \mathbb{D}_{\frac{11}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}(\frac{11}{2}) + \mathbb{D}_{\frac{11}{2},a}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}(\frac{11}{2}),a \\ &+ \sum_{I=a,b} \mathbb{D}_{6,I}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(6),I} + \mathbb{D}_{\frac{13}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(\frac{13}{2})} + \mathbb{D}_{\frac{13}{2},a}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(\frac{13}{2}),a} + \sum_{I=a,b} \mathbb{D}_{7,I}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(7),I} \\ &+ \mathbb{D}_{\frac{15}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(\frac{15}{2})} + \sum_{I=a,b,c} \mathbb{D}_{\frac{15}{2},I}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(\frac{15}{2}),I} \\ &+ \sum_{I=a,\dots,f} \mathbb{D}_{8,I}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(8),I} + \mathbb{D}_{\frac{17}{2}}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(\frac{17}{2})} + \sum_{I=a,\dots,e} \mathbb{D}_{8,I}^{\frac{7}{2},\frac{11}{2}} \mathbb{V}^{(\frac{17}{2}),I}. \end{aligned} \quad (\text{B.15})$$

Imposing the associativity of $\mathbb{V}(\frac{7}{2}) \times \mathbb{V}(\frac{7}{2}) \times \mathbb{V}(\frac{7}{2})$, the various couplings turn out to equal

$$\mathbb{D}_{\frac{7}{2}}^{\frac{7}{2},\frac{11}{2}} = \frac{13824(4c+21)(6c-13)(605c^2+1278c-8456)}{11c^2(2c+37)(2c+53)(10c-7)^2}, \quad (\text{B.16})$$

$$\mathbb{D}_{\frac{11}{2}}^{\frac{7}{2},\frac{11}{2}} = \frac{336(2288c^4+47930c^3+189489c^2-2153000c-4668092)}{11c(2c+37)(2c+53)(4c+21)(10c-7)}, \quad (\text{B.17})$$

$$\mathbb{D}_{6,a}^{\frac{7}{2},\frac{11}{2}} = -\frac{829440(14c+11)(502c^2+8203c-28424)}{121c^2(2c+37)(2c+53)(10c-7)^2}, \quad (\text{B.18})$$

$$\mathbb{D}_{8,a}^{\frac{7}{2},\frac{11}{2}} = \text{complicated rational function of } c, \quad (\text{B.19})$$

$$\mathbb{D}_{\frac{11}{2},a}^{\frac{7}{2},\frac{11}{2}} = \frac{46448640(c+11)(6c-13)(7c-10)(23c-11)}{121c^2(2c+37)^2(2c+53)(10c-7)^2}, \quad (\text{B.20})$$

$$\mathbb{D}_{7,b}^{\frac{7}{2},\frac{11}{2}} = \frac{20736(26c-77)}{11c(2c+37)(10c-7)}, \quad (\text{B.21})$$

$$\mathbb{D}_{8,d}^{\frac{7}{2},\frac{11}{2}} = \frac{64(58924c^2 + 963796c + 2518087)}{55c(2c+37)(2c+53)(10c-7)(13c+5)}, \quad (\text{B.22})$$

$$\mathbb{D}_{\frac{17}{2},d}^{\frac{7}{2},\frac{11}{2}} = \frac{24300(26c-77)}{11c(2c+37)(5c+77)(10c-7)}, \quad (\text{B.23})$$

$$\mathbb{D}_{8,e}^{\frac{7}{2},\frac{11}{2}} = -\frac{32(5024c^3 + 266906c^2 + 1815131c + 1654059)}{55c(2c+37)(2c+53)(10c-7)(13c+5)}. \quad (\text{B.24})$$

Furthermore, the other couplings vanish

$$\begin{aligned} \mathbb{D}_{\frac{5}{2}}^{\frac{7}{2},\frac{11}{2}} &= \mathbb{D}_{\frac{9}{2}}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{13}{2}}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{17}{2}}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{7,a}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{13}{2},a}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{17}{2},a}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{6,b}^{\frac{7}{2},\frac{11}{2}} \\ &= \mathbb{D}_{8,b}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{17}{2},b}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{8,c}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{17}{2},c}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{\frac{17}{2},e}^{\frac{7}{2},\frac{11}{2}} = \mathbb{D}_{8,f}^{\frac{7}{2},\frac{11}{2}} = 0. \end{aligned} \quad (\text{B.25})$$

The choice of basis for the various composite superprimaries is largely arbitrary. We determined them automatically, and as a consequence some of the coupling constants are complicated rational functions, e.g. the one appearing in $\mathbb{D}_{8,a}^{\frac{7}{2},\frac{11}{2}}$.

B.4 The OPEs $\frac{9}{2} \times \frac{9}{2}$ and $\frac{5}{2} \times \frac{13}{2}$

We have also made the most general ansatz for the OPEs $\mathbb{V}^{(\frac{9}{2})} \times \mathbb{V}^{(\frac{9}{2})}$ and $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{13}{2})}$

$$\begin{aligned} \mathbb{V}^{(\frac{9}{2})} \times \mathbb{V}^{(\frac{9}{2})} &= \mathbb{D}_0^{\frac{9}{2},\frac{9}{2}} \mathbb{I} + \mathbb{D}_{\frac{7}{2}}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(\frac{7}{2})} + \mathbb{D}_{\frac{11}{2}}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(\frac{11}{2})} + \mathbb{D}_{\frac{11}{2},a}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(\frac{11}{2},a)} + \sum_{I=a,b} \mathbb{D}_{6,I}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(6),I} \\ &\quad + \mathbb{D}_{\frac{15}{2}}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(\frac{15}{2})} + \sum_{I=a,b,c} \mathbb{D}_{\frac{15}{2},I}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(\frac{15}{2}),I} + \sum_{I=a,\dots,f} \mathbb{D}_{8,I}^{\frac{9}{2},\frac{9}{2}} \mathbb{V}^{(8),I}, \end{aligned} \quad (\text{B.26})$$

$$\begin{aligned} \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{13}{2})} &= \mathbb{D}_{\frac{7}{2}}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{7}{2})} + \mathbb{D}_{\frac{9}{2}}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{9}{2})} + \mathbb{D}_{\frac{11}{2}}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{11}{2})} + \mathbb{D}_{\frac{11}{2},a}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{11}{2},a)} + \sum_{I=a,b} \mathbb{D}_{6,I}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(6),I} \\ &\quad + \sum_{I=a,b} \mathbb{D}_{7,I}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(7),I} + \mathbb{D}_{\frac{15}{2}}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{15}{2})} + \sum_{I=a,b,c} \mathbb{D}_{\frac{15}{2},I}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{15}{2}),I} \\ &\quad + \sum_{I=a,\dots,f} \mathbb{D}_{8,I}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(8),I} + \mathbb{D}_{\frac{17}{2}}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{17}{2})} + \sum_{I=a,\dots,e} \mathbb{D}_{8,I}^{\frac{5}{2},\frac{13}{2}} \mathbb{V}^{(\frac{17}{2}),I}. \end{aligned} \quad (\text{B.27})$$

The associativity of $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{7}{2})} \times \mathbb{V}^{(\frac{9}{2})}$ and $\mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{5}{2})} \times \mathbb{V}^{(\frac{11}{2})}$ can then be imposed as in the previous steps. We do not write the detailed list of couplings that are fixed in this way, except for giving the specific case of

$$\mathbb{D}_0^{\frac{9}{2},\frac{9}{2}} = \frac{28800(2c+5)(6c-13)(7c-10)}{(7-10c)^2c^2(2c+37)}, \quad (\text{B.28})$$

that will play a role for the truncation analysis of Section 4.4.

C Normalisation of the composite superprimaries

In order to fix our conventions, we report here the leading contributions to the composite superprimaries; the remaining terms that are needed in order to make them superprimaries are G and T descendants of fields of lower conformal dimension. We also only specify the

‘top’ component of the superprimary. For $\mathbb{V}^{(\frac{11}{2}),a}$, $\mathbb{V}^{(6),a}$, $\mathbb{V}^{(6),b}$ and $\mathbb{V}^{(\frac{13}{2}),a}$ the relevant expressions were already given after (4.9) and in (4.16). The other cases of interest to us are

$$V^{(7),a} = \frac{1}{5} (V^{(3)} V^{(4)}) + \dots, \quad (C.1)$$

$$V^{(7),b} = (V^{(\frac{5}{2})} V^{(\frac{9}{2})}) + \dots, \quad (C.2)$$

$$\begin{aligned} V^{(\frac{15}{2}),a} = & -\frac{(2c^2 + 151c + 2166) (V^{(\frac{5}{2})} V^{(3)'})}{6(25c + 956)} + \frac{(14c^2 + 673c + 6510) (V^{(\frac{5}{2})}' V^{(3)'})}{15(25c + 956)} \\ & - \frac{(14c^2 + 757c + 3690) (V^{(\frac{5}{2})} V^{(3)'})}{30(25c + 956)} \\ & + \frac{(240c^4 + 31436c^3 + 1137640c^2 + 13454979c + 21528990) V^{(\frac{7}{2})''''}}{960(25c + 956) (20c^2 + 708c + 1197)} + \dots, \end{aligned} \quad (C.3)$$

$$V^{(\frac{15}{2}),b} = (V^{(\frac{7}{2})} V^{(4)}) + \dots, \quad (C.4)$$

$$\begin{aligned} V^{(\frac{15}{2}),c} = & \frac{2(4c + 31)(V^{(3)} V^{(\frac{7}{2})}')}{5(c + 20)} - \frac{2(4c + 31)(V^{(\frac{5}{2})}' V^{(4)})}{5(c + 20)} - \frac{14(2c + 33)(V^{(3)'} V^{(\frac{7}{2})})}{15(c + 20)} \\ & + (V^{(\frac{5}{2})} V^{(4)'}) - \frac{32(10c^4 - 421c^3 - 3317c^2 + 26691c + 10112) V^{(\frac{5}{2})''''}}{25c(c + 20)(2c + 37)(4c + 21)(10c - 7)} \\ & - \frac{(2c + 19)V^{(\frac{9}{2})'''}}{15(c + 20)} + \dots, \end{aligned} \quad (C.5)$$

$$V^{(\frac{15}{2}),d} = (V^{(\frac{5}{2})} V^{(5)}) + \frac{9}{5} (V^{(3)} V^{(\frac{9}{2})}) \dots. \quad (C.6)$$

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